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Quantum disordering and the phase diagram of bilayer Josephson junction arrays

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Abstract

I present a theoretical analysis of the phase diagram of bilayer Josephson junction arrays in the presence of charge and magnetic frustration. Using a dual description I show that the fundamental constituents of this theory are electric and magnetic excitations and that their condensations lead to a plethora of possible phases. The new formulation points out to the emergence of a rich phase diagram not attainable by standard mean field theory approaches. In addition to the usual superconducting and insulating states in the bilayer system, I find states exhibiting Hall quantization coexisting with interlayer coherence, states with Hall quantization without interlayer coherence and interlayer coherent states without Hall quantization.

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1. Introduction

The phase diagram of Josephson junction arrays (JJA) has a rich structure with many prominent features such as an insulator–superconductor quantum phase transition [1] and topological orders. A gauge field theory description of JJA was formulated in [2], which provided a natural framework to study the manifestation of topological orders in these systems. Starting from the quantum phase model which captures the relevant physics of JJA, the model was mapped, in the self-dual approximation, into an Abelian gauge theory with a mixed Chern–Simons term. Together with duality, this topological gauge theory allowed for the study of topological order and the phase diagram of these systems. Furthermore in the presence of frustration due to offset charges and external magnetic flux quanta, several theoretical investigations suggested that JJA might support incompressible quantum fluid phases corresponding to purely two-dimensional quantum Hall phases for either charges or vortices [3].

In [4] I re-examined self-dual JJA systems in the framework of a new Landau–Ginzburg theory. The low energy dynamics was shown to be governed by two complex fields associated with disordering due to electric charges and magnetic charges and minimally coupled to two gauge fields which determine the currents of Cooper pairs and vortices. I showed that quantum disordering [5] due to topological electric and magnetic excitations describes effectively various phases of these systems. In the new formulation, the superconducting, insulating and Hall phases emerge from the condensation of composite fields which describe bound states of electric and magnetic topological excitations. Contrary to previous studies of the Hall states in JJA [3], the approach in [4] did not rely on the addition of any Chern–Simons term.

In this paper, I wish to further analyze the properties of coupled bilayer Josephson junction arrays which present a rich phase diagram due to extra degrees of freedom associated with the layer index and interlayer coupling. I construct a dual description whose fundamental constituents are topological electric and magnetic excitations and I show that this formalism leads to a plethora of possible phases. Some of these phases describe incompressible quantum Hall states with or without interlayer coherence. The results obtained are significant since they imply the realization of bosonic quantized Hall states. The paper is organized as follows. In section 2 I give the basic ideas of the new Landau–Ginzburg theory and its gauge theory representation in the context of one single-layer JJA system, then in section 3 I apply these ideas to the more interesting case of bilayer JJA, and I address systematically the electromagnetic response and the quantum Hall phenomenology of these systems. The relevance of the present results is discussed in the concluding paragraphs of this paper.

2. Single-layer JJA system

A Josephson junction array consists of a regular network of superconducting islands weakly coupled by tunnel junctions. Each junction is characterized by a Josephson coupling E_J and a junction capacitance C. Here I consider the limit where the nearest-neighbors capacitance C dominates the capacitance to the ground. I include an external magnetic field B_{ext} to induce vortices in the system, and I allow for offset charges on the superconducting grains $Q_x = 2e\overline{n}_q$. The quantum phase model Hamiltonian describing the charges and vortices in this system [1] is given by

$$\widehat{H} = \frac{4e^2}{2}\sum_{\langle ij\rangle}(n_i - \overline{n}_q)C_{ij}^{-1}(n_j - \overline{n}_q) - E_J\sum_{\langle ij\rangle}\cos(\phi_i - \phi_j - A_{ij}). \qquad (1)$$

The first term in the Hamiltonian is the charging energy in which the C_{ij}^{-1} is the inverse capacitance matrix ($C_{ii} = 4C$ and $C_{ij} = -C$ for nearest neighbors); the second term is due to the Josephson tunneling. The variables n_i and ϕ_i denote respectively the charge and the phase of the superconducting order parameter of the ith island in the array, and the sums are over nearest neighbors. The competition between these two canonically conjugated variables is captured by the usual commutation relation $[n_i, \phi_j] = \mathrm{i}\delta_{ij}$. A perpendicular magnetic field with vector potential A enters the Hamiltonian through $A_{ij} = 2e\int_i^j A\cdot \mathrm{d}l$.

This Hamiltonian is mapped into an effective Abelian gauge theory [2] formulated in terms of two gauge fields a_μ and b_μ, which are related to two conserved currents $p_\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda/(2\pi)$, $\mu = 0, 1, 2$ and $l_\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/(2\pi)$ associated with the motion of charges and vortices. The dynamics of these fluctuations in the self-dual approximation [2] is governed by the imaginary-time Lagrangian

$$L = \frac{\kappa_1}{4}f_{\mu\nu}^2 + \frac{\kappa_2}{4}g_{\mu\nu}^2 + \frac{\mathrm{i}}{2\pi}(b_\mu + \overline{b}_\mu)\varepsilon^{\mu\nu\lambda}\partial_\nu(a_\lambda + \overline{a}_\lambda) + \mathrm{i}a_\mu Q_\mu + \mathrm{i}b_\mu M_\mu, \qquad (2)$$

where $f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$, $g_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and the background gauge potentials $\overline{a}$ and $\overline{b}$, defined by $\overline{a}_o = \overline{b}_o = 0$, $\partial_o\overline{a}_j = \partial_o\overline{b}_j = 0$, $\vec{\nabla}\wedge\overline{a} = -2eB_{\mathrm{ext}}$, $\vec{\nabla}\wedge\overline{b} = 2\pi\overline{n}_q$, account for frustration due to external magnetic fields and to offset charges. The coupling constants κ_1 and κ_2 are related to the Josephson energy and to the charging energy $\kappa_1 = 1/(4\pi^2 E_J), \kappa_2 = 1/(8E_c)$. Equation (2) shows that the moving particles associated with the current M_i see a 'magnetic field' $\varepsilon^{ij}\partial_i(\overline{b}_j + b_j)/2\pi$ equal to the sum of the density of bosons and a fluctuating field. Similarly, the moving particles associated with the current Q_i see a 'magnetic field' $\varepsilon^{ij}\partial_i(\overline{a}_j + a_j)/2\pi$ equal to the sum of the density of fluxes and a fluctuating field.

Using standard $U(1)$ particle–vortex duality [6], complex scalar fields are introduced to create and annihilate the topological excitations Q_μ and M_μ by elaborating the description in equation (2) to a Landau–Ginzburg theory as shown in [4]. This dual Landau–Ginzburg representation in terms of two scalar fields ϕ_C and ϕ_M is convenient to study JJA systems since its phase structure can be analyzed by considering the condensation of these fields. More generally, one can consider a generic condensate consisting of n quasiparticles of type ϕ_C and m quasiparticles of type ϕ_M. Such fields, $\phi_{n,m}$, carry n units of the a_μ charge and m units of the b_μ charge, and are described by the following Euclidean effective theory:

$$L = \frac{1}{2}|(\partial_\mu - \mathrm{i}n(a_\mu - \overline{a}_\mu) - \mathrm{i}m(b_\mu - \overline{b}_\mu))\phi_{n,m}|^2 + V(\phi_{n,m}) + \frac{\mathrm{i}}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{\kappa_1}{4}f_{\mu\nu}^2 + \frac{\kappa_2}{4}g_{\mu\nu}^2 + \mathrm{i}\frac{2e}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{3}$$

The potential $V(f)$ can be expanded as $V(f) = rf^2 + uf^4 + \cdots$, and describes the short distance physics contained in the original theory defined on the lattice. The phase diagram is analyzed by considering the condensation of the fields $\phi_{n,m}$. A phase in which the topological charge excitations are absent corresponds to taking r large and positive in the potential $V(f)$, while a phase in which these degrees proliferate and condense, $\langle\phi_{n,m}\rangle \neq 0$, corresponds to taking r negative. The latter phase is assumed to occur when the background fields cancel. The last term in equation (3) couples the current charges with external probing electromagnetic field A_μ.

Before we proceed with the analysis of the model defined in (3), some comments about the significance and restriction of the numbers n and m are in order. On one hand the reduced equations of motion obtained by varying the Lagrangian with respect to the gauge fields a_μ and b_μ yield the following relation for the current associated with the condensate $J_\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda/(\pi n) = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/(\pi m)$, which implies that each composite carries two fluxes $\phi_a = m\pi$ and $\phi_b = n\pi$ respectively with the gauge fields a_μ and b_μ. On the other hand, at a microscopic level, the effect of a flux tube on particles is to add a phase factor proportional to the winding multiplying the amplitude for trajectories where these objects wind around each other. This alters the spectrum of allowed angular momentum, which in the present case is $L = nm$. The phase acquired by these objects as they are slowly interchanged at great distances is a measure of their quantum statistics [7], and here it is equal to $\theta = \pi nm$. Since in JJA the charge degrees of freedom are bosons (Cooper pairs), the fields $\phi_{n,m}$ must also have bosonic statistics, i.e. nm even integer. Furthermore, using the field equations, the coupling with an external electromagnetic field is given by a term in the Lagrangian $\Delta L = 2enA_\mu J_\mu$, which shows that the condensate $\phi_{n,m}$ carries a physical charge $2en$. Since the fundamental electric charges in JJA are Cooper pairs with charge $2e$, in our analysis of the various phases we restrict n to 0 or 1 and allow m to take even integer values.

All the thermodynamic properties as well as the electromagnetic response of the system are completely determined from the path integral

$$Z[A_\mu] = \int [\mathrm{d}a_\mu][\mathrm{d}b_\mu][\mathrm{d}\phi]\exp\left(-\int \mathrm{d}^3x L\right). \tag{4}$$

The technique we adopt in evaluating the partition function $Z[A_\mu]$ is to linearize terms in the Lagrangian so that Gaussian integrations with respect to quadratic terms in the gauge fields can be easily carried out [8].

In this paper we provide more details about the dual formulation of the theory defined by equation (3) which later will be generalized to the bilayer system. We rewrite the Higgs field as $\phi = f\mathrm{e}^{\mathrm{i}\omega}$ with the phase split into multivalued and regular parts $\omega = \overline{\omega} + \chi$ where the first term describes a configuration of vortices of ϕ and χ represents single-valued fluctuations around such configurations

$$L = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}f^2[\partial_\mu\omega - n(a_\mu - \overline{a}_\mu) - m(b_\mu - \overline{b}_\mu)]^2 + V(f) + \frac{\mathrm{i}}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{\kappa_1}{4}f_{\mu\nu}^2 + \frac{\kappa_2}{4}g_{\mu\nu}^2 + \mathrm{i}\frac{2e}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{5}$$

To perform the path integral (4), we first linearize the second term of the Lagrangian (5) by introducing an auxiliary field C_μ

$$L = \frac{1}{2}(\partial_\mu f)^2 + \mathrm{i}C_\mu[\partial_\mu\chi + \partial_\mu\overline{\omega} - n(a_\mu - \overline{a}_\mu) - m(b_\mu - \overline{b}_\mu)] + \frac{1}{2f^2}(C_\mu)^2 + V(f) + \frac{\mathrm{i}}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{\kappa_1}{4}f_{\mu\nu}^2 + \frac{\kappa_2}{4}g_{\mu\nu}^2 + \mathrm{i}\frac{2e}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{6}$$

As χ is single-valued, one can integrate it out leading to $\partial_\mu C_\mu = 0$. This constraint is solved by introducing a dual gauge field: $C^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu H_\lambda$ and the resulting Lagrangian is

$$L = \frac{1}{2}(\partial_\mu f)^2 + \mathrm{i}[n(a_\mu - \overline{a}_\mu) + m(b_\mu - \overline{b}_\mu)]\varepsilon^{\mu\nu\lambda}\partial_\nu H_\lambda + \frac{1}{4f^2}(H_{\mu\nu})^2 + V(f) + \frac{\mathrm{i}}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{\kappa_1}{2}f_{\mu\nu}^2 + \frac{\kappa_2}{2}g_{\mu\nu}^2 + \mathrm{i}\frac{2e}{\pi}b_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{7}$$

To analyze this effective action, I diagonalize the gauge part of the Lagrangian by using the linear transformation $a_\mu = (X_\mu + Y_\mu)\sqrt[4]{\kappa_1/\kappa_2}$, $b_\mu = (X_\mu - Y_\mu)\sqrt[4]{\kappa_2/\kappa_1}$

$$L = \frac{1}{2}(\partial_\mu f)^2 + V(f) + \frac{1}{4\pi\omega_J}\left(X_{\mu\nu}^2 + Y_{\mu\nu}^2\right) + \mathrm{i}\eta X_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu X_\lambda - \mathrm{i}\eta Y_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu Y_\lambda + \frac{1}{4f^2}H_{\mu\nu}^2 + \mathrm{i}(\theta_1 X_\mu + \theta_2 Y_\mu)\varepsilon^{\mu\nu\lambda}\partial_\nu H_\lambda + \mathrm{i}(X_\mu - Y_\mu)\varepsilon^{\mu\nu\lambda}\partial_\nu\widetilde{A}_\lambda \tag{8}$$

where $\eta = 1/\pi$, $\theta_{1(2)} = n\sqrt[4]{\kappa_1/\kappa_2} \pm m\sqrt[4]{\kappa_2/\kappa_1}$, $2\pi\omega_J = 1/\sqrt{\kappa_1\kappa_2}$ and $\widetilde{A}_\lambda = \frac{2e}{\pi}\sqrt[4]{\kappa_2/\kappa_1}A_\lambda$. To achieve an effective action for the probing gauge field $\widetilde{A}_\lambda$ we integrate out all gauge fields X_μ, Y_μ and H_μ by choosing a suitable gauge fixing condition. At an intermediate step we find

$$L = \frac{1}{2}H_\mu D_{\mu\nu}^{-1}H_\nu + \frac{1}{2}\Pi_1\widetilde{F}_{\mu\nu}^2 + \frac{1}{2}(\theta_1 - \theta_2)\Pi_1 H_{\mu\nu}\widetilde{F}_{\mu\nu} - \mathrm{i}\pi\omega_J\eta(\theta_1 + \theta_2)\Pi_1 H_\mu\varepsilon^{\mu\nu\lambda}\widetilde{F}_{\nu\lambda}, \tag{9}$$

where in momentum space $D_{\mu\nu}^{-1}(q) = \left[1/f^2 + \Pi_1\left(\theta_1^2 + \theta_2^2\right)\right](q^2\delta_{\mu\nu} - q_\mu q_\nu) + 2\pi\omega_J\eta\left(\theta_2^2 - \theta_1^2\right)\Pi_1\varepsilon^{\mu\lambda\nu}q_\lambda$ and $\Pi_1 = \pi\omega_J/[q^2 + (2\pi\omega_J\eta)^2]$. Next integration over H_μ gives an effective action for the probing gauge field $\widetilde{A}_\lambda$

$$S_g = \int q^2\Pi_1\Pi_4[q^2/f^2 + \pi\omega_J(\theta_1 + \theta_2)^2/2]\widetilde{A}_\mu\varepsilon^{\mu\lambda\nu}q_\lambda\widetilde{A}_\nu + \frac{1}{2}\widetilde{A}_\mu(q^2\delta_{\mu\nu} - q_\mu q_\nu)\widetilde{A}_\nu \times\left[2\Pi_1 - \Pi_1^2\Pi_3\{q^2(\theta_1 - \theta_2)^2 - (2\pi\omega_J\eta)^2(\theta_1 + \theta_2)^2\} - 2q^2\Pi_1^2\Pi_4(2\pi\omega_J\eta)\left(\theta_1^2 - \theta_2^2\right)\right], \tag{10}$$

where $\Pi_3^{-1} = q^2/\xi^2 + \sigma^2\xi^2$, $q^2\Pi_4 = -\Pi_3\sigma\xi^2$, $\sigma = (2\pi\omega_J\eta)\left(\theta_2^2 - \theta_1^2\right)\Pi_1$ and $1/\xi^2 = 1/f^2 + \Pi_1\left(\theta_1^2 + \theta_2^2\right)$. This action is the central result of this section, it encodes all the information about the phenomenology of the one-layer JJA system. The electromagnetic response of the system is derived from the correlation functions $\delta^2 S/\delta A_\mu(-q)\delta A_\nu(q)$.

2.1. Electric condensation

In this phase $n = 1$ and $m = 0$. Working in zero background magnetic field and using $\theta_1 = \theta_2 = \sqrt[4]{\kappa_1/\kappa_2}$ we find that the coefficient of the Chern–Simons term vanishes. The long distance physics is determined by a Meissner action

$$S_g = \int \frac{1}{2} A_\mu \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) A_\nu \frac{4e^2 f^2}{1 + 4\pi^2 f^2 \kappa_1}. \tag{11}$$

The induced electromagnetic current obtained by varying the action, $J_\mu = \delta S / \delta A_\mu$ gives the standard London form. This shows that the electric condensation phase is actually a superconducting phase. In this phase the vortices are confined and their mass $m_\phi = 1/4E_C$ is much larger than that of the electric charges $m_q = 1/(2\pi^2 E_J)$, hence this phase is realized for $\kappa_1 \ll \kappa_2$.

2.2. Magnetic condensation

In this phase $n = 0, m \neq 0$. Working in zero background offset charges and using $\theta_1 = -\theta_2 = m\sqrt[4]{\kappa_2/\kappa_1}$ the coefficient of the Chern–Simons term vanishes and the long distance physics is determined purely by the Maxwell term

$$S_g = \frac{1}{2} \int A_\mu (q^2 \delta_{\mu\nu} - q_\mu q_\nu) A_\nu \frac{4e^2 \kappa_2}{1 + 4\pi^2 f^2 m^2 \kappa_2}. \tag{12}$$

The gauge field A_μ is massless and the longitudinal conductivity vanishes. This shows that the magnetic condensation corresponds to an insulating phase. In this phase the charges are confined and their mass is much heavier $m_q \gg m_\phi$, hence this phase is realized for $\kappa_1 \gg \kappa_2$.

As expected the magnetic and the electric condensations are symmetric around the point $\kappa_1 = \kappa_2$ reflecting the self-duality of the model.

2.3. Condensation of electric-magnetic bound states

In this phase we take $n = 1$ and m is an even integer. To obtain condensation of composite fields in zero background field we require that $eB_{\text{ext}}/\pi = m\overline{n}_q$. The effective dynamics of the external electromagnetic gauge potential has a non-vanishing Chern–Simons term in addition to the Maxwell term

$$S_g = \frac{e^2}{\pi m} \int A_\mu \varepsilon^{\mu\lambda\nu} q_\lambda A_\nu + \frac{1}{8m^2 \xi^2} \int A_\mu (q^2 \delta_{\mu\nu} - q_\mu q_\nu) A_\nu. \tag{13}$$

Since the latter term is higher dimensional, it is less important at long distance and the remaining Chern–Simons describes a Hall fluid. Varying with respect to A_μ we determine the electromagnetic current $J_\mu = 2e^2/(\pi m)\varepsilon^{\mu\lambda\nu}\partial_\lambda A_\nu$. From the $\mu = 0$ component of the current we see that an excess δn of bound charges is related to a local fluctuation of the magnetic field by $\delta n = en/(\pi m)\delta B$, which allows us to identify the filling factor $\nu = 1/m$. From the $\mu = \mathrm{i}$ component of the current, an electric field produces a current in the orthogonal direction with a Hall conductivity $\sigma_{xy} = 1/m$ in units of $4e^2/(2\pi\hbar)$.

3. Bilayer JJA

In this section, we consider a quasi-three-dimensional array composed of two coupled JJA layers (see figure 1). An appealing feature of coupling two layers of JJA is the possibility to tune independently the interlayer capacitance and thereby control the interaction between

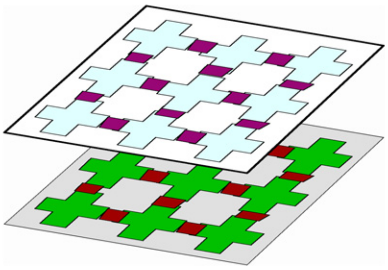

Figure 1. Schematic representation of a bilayer Josephson junction array.

vortices and charges. As a result, depending on the arrays' physical parameters, each layer can be in a regime dominated by Cooper pairs (superfluid) or a regime dominated by vortices (insulating). This system has been investigated in [9] with the aim of showing the existence of a duality between charges and vortices. The focus in [9] was on the situation when one array is in the quasi-classical (vortex) regime while the other is in the quantum (charge) regime. The resulting effective action describes dual charges in one array and vortices in the other, and in contrast to the one-layer problem, these are dynamic degrees of freedom.

A mean field theory description was used in [10] to study the changes in the individual critical temperature of two JJA, when both were in the semiclassical parameter regime. Using a WKB semiclassical expansion valid for $E_C \ll E_J$, an effective Hamiltonian was analyzed within a variational mean field theory. The evaluated critical temperature shift showed that an increase in the interaction capacitance increases phase coherence in the arrays. In the case when one array is quantum phase dominated and the other Cooper pair charge dominated, the derived effective Hamiltonian is dually symmetric between charges and vortices, and exhibits in the simplified case where one array has one vortex and the other one charge, a gauge-like interaction, implying that a vortex feels an effective magnetic field produced by the charge. However, that mean field theory did not allow for a more elaborate study of the interplay of quantum-classical effects resulting from the vortex–charge interaction. In particular there was no discussion about the quantum Hall phases caused by charge–vortex bound states.

The formalism employed in this paper is more suitable to directly study the gauge-like interaction between charges and vortices with the additional advantage of predicting a variety of exotic charge–vortex bound states not attained by the mean field theory of [10]. For the bilayer JJA system, the model is formulated in terms of four gauge fields $a_\mu^{(\alpha)}$ and $b_\mu^{(\alpha)}$ with $\alpha = 1, 2$ the layer index. These fields describe the conserved currents of charges $(1/2\pi)\varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda^{(\alpha)}$, and the conserved currents of vortices $(1/2\pi)\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(\alpha)}$.

Imposing self-duality allows a suitable gauge theory representation of the model of interacting charges and vortices, and contrary to previous studies, it generates in the bilayer JJA system the most general interactions between the charges and the vortices [4]. The model has intralayer and interlayer electric interactions similar to those introduced in [8], as well as intralayer and interlayer vortex–vortex interactions that emerge from the imposed high degree of symmetry. The ensuing dynamics is rich and leads to a large class of possible states. The dynamics is governed by the imaginary-time Lagrangian

$$L = \frac{\kappa_1}{4}\left[f_{\mu\nu}^{(\alpha)}\right]^2 + \frac{\kappa_2}{4}\left[g_{\mu\nu}^{(\alpha)}\right]^2 + \mathrm{i}\eta_{\alpha\beta}\left(b_\mu^{(\alpha)} + \overline{b}_\mu^{(\alpha)}\right)\varepsilon^{\mu\nu\lambda}\partial_\nu\left(a_\lambda^{(\beta)} + \overline{a}_\lambda^{(\beta)}\right) + \mathrm{i}a_\mu^{(\alpha)}Q_\mu^{(\alpha)} + \mathrm{i}b_\mu^{(\alpha)}M_\mu^{(\alpha)}. \quad (14)$$

The matrix $\widehat{\eta}$ appearing in equation(13) has elements in momentum space given by: $\eta_{11} = \eta_{22} = (1 + q/\sqrt{q^2+\Lambda^2})/(4\pi)$, $\eta_{12} = (1 - q/\sqrt{q^2+\Lambda^2})/(4\pi)$ and $\Lambda^2 = 2C_I/C$

with C_I is the interlayer capacitance between each island in one array coupled parallel to one island in the other array (straight coupling) and C the nearest-neighbors capacitance.

The same approach used before in the one-layer case can also be used to construct the effective field theory of the bilayer JJA system. More generally one considers composite condensates consisting of bound states of $n^{(\alpha)}$ excitations in $\phi_C^{(\alpha)}$ and $m^{(\alpha)}$ excitations in $\phi_M^{(\alpha)}, \phi_{\{n,m\}} \sim \phi_C^{n^{(1)}} \phi_C^{n^{(2)}} \phi_M^{m^{(1)}} \phi_M^{m^{(2)}}$, where $n^{(\alpha)}$ and $m^{(\alpha)}$ are integers. The Lagrangian that needs to be analyzed is

$$L = \frac{1}{2}(\partial_\mu f)^2 + \mathrm{i}\big[n_\alpha\big(a_\mu^{(\alpha)} - \overline{a}_\mu\big) + m_\alpha\big(b_\mu^{(\alpha)} - \overline{b}_\mu\big)\big]\varepsilon^{\mu\nu\lambda}\partial_\nu H_\lambda + \frac{1}{4f^2}(H_{\mu\nu})^2 + V(f) + \mathrm{i}\eta_{\alpha\beta} b_\mu^{(\alpha)}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(\beta)} + \frac{\kappa_1}{4}\big[f_{\mu\nu}^{(\alpha)}\big]^2 + \frac{\kappa_2}{4}\big[g_{\mu\nu}^{(\alpha)}\big]^2 + \mathrm{i}\frac{e}{\pi} b_\mu^{(\alpha)}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^{(\alpha)}, \tag{15}$$

where as before f is the amplitude of the condensate and the gauge field H_μ results from integrating over the phase of the condensate. To analyze this effective action, I diagonalize the gauge part of the Lagrangian by writing $a_\mu^{(\alpha)} = \big(X_\mu^{(\alpha)} + Y_\mu^{(\alpha)}\big)\sqrt[4]{\kappa_1/\kappa_2}$, $b_\mu^{(\alpha)} = \big(X_\mu^{(\alpha)} - Y_\mu^{(\alpha)}\big)\sqrt[4]{\kappa_2/\kappa_1}$ and by using the symmetric and antisymmetric combinations $X_\mu^{\pm} = \big(X_\mu^{(1)} \pm X_\mu^{(2)}\big)/\sqrt{2}$, $Y_\mu^{\pm} = \big(Y_\mu^{(1)} \pm Y_\mu^{(2)}\big)/\sqrt{2}$. In terms of these new fields, the gauge part of the Lagrangian is

$$L = \frac{1}{4f^2}H_{\mu\nu}^2 + \sum_{\sigma=\pm}\frac{1}{4\pi\omega_J}\left(X_{\mu\nu}^{(\sigma)2} + Y_{\mu\nu}^{(\sigma)2}\right) + \mathrm{i}\eta^{(\sigma)}X_\mu^{(\sigma)}\varepsilon^{\mu\nu\lambda}\partial_\nu X_\lambda^{(\sigma)} - \mathrm{i}\eta^{(\sigma)}Y_\mu^{(\sigma)}\varepsilon^{\mu\nu\lambda}\partial_\nu Y_\lambda^{(\sigma)} + \mathrm{i}\big(\theta_1^{(\sigma)}X_\mu^{(\sigma)} + \theta_2^{(\sigma)}Y_\mu^{(\sigma)}\big)\varepsilon^{\mu\nu\lambda}\partial_\nu H_\lambda + \big(X_\mu^{(\sigma)} - Y_\mu^{(\sigma)}\big)\varepsilon^{\mu\nu\lambda}\partial_\nu \widetilde{A}_\lambda^{(\sigma)}. \tag{16}$$

To obtain the effective field theory of the probing electromagnetic field, we integrate out the gauge fields $X_\mu^{(\sigma)}, Y_\mu^{(\sigma)}$ and H_μ using suitable gauge fixing conditions. This can be easily achieved since the action is quadratic in these fields and the result is

$$S = \frac{1}{2}\int\sum_{\sigma,\sigma'=\pm}\widetilde{A}_\mu^{(\sigma)}(q^2\delta_{\mu\nu} - q_\mu q_\nu)\widetilde{A}_\nu^{(\sigma')}\big[2\Pi_1^\sigma\delta_{\sigma,\sigma'} - \Pi_3\Gamma^{\sigma\sigma'} - q^2\Pi_4\Delta^{\sigma\sigma'}\big] - \frac{1}{2}\int\sum_{\sigma,\sigma'=\pm}\widetilde{A}_\mu^{(\sigma)}\varepsilon^{\mu\lambda\nu}q_\lambda\widetilde{A}_\nu^{(\sigma')}[q^2\Pi_4\Gamma^{\sigma\sigma'} - q^2\Pi_3\Delta^{\sigma\sigma'}], \tag{17}$$

where $\Gamma^{\sigma\sigma'} = x^{(\sigma)}x^{(\sigma')}q^2 - y^{(\sigma)}y^{(\sigma')}$; $\Delta^{\sigma\sigma'} = x^{(\sigma)}y^{(\sigma')} + y^{(\sigma)}x^{(\sigma')}$; $x^{(\sigma)} = \Pi_1^\sigma\big(\theta_1^{(\sigma)} - \theta_2^{(\sigma)}\big)$; $y^{(\sigma)} = (2\pi\omega_J\eta^{(\sigma)})\Pi_1^\sigma\big(\theta_1^{(\sigma)} + \theta_2^{(\sigma)}\big)$; $\theta_{1(2)}^\sigma = n^\sigma\sqrt[4]{\kappa_1/\kappa_2} \pm m^\sigma\sqrt[4]{\kappa_2/\kappa_1}$,

$\Pi_1^\sigma(q) = \pi\omega_J/[q^2 + (2\pi\omega_J\eta^\sigma)^2]$; $\Pi_3^{-1} = q^2/\rho^2 + \Theta^2\rho^2$; $q^2\Pi_4 = -\Pi_3\Theta\rho^2$; $\Theta = \sum_{\sigma=\pm}(2\pi\omega_J\eta^\sigma)\big(\theta_2^{\sigma2} - \theta_1^{\sigma2}\big)\Pi_1^\sigma$, and $1/\rho^2 = 1/f^2 + \sum_{\sigma=\pm}\Pi_1^\sigma\big(\theta_1^{\sigma2} + \theta_2^{\sigma2}\big)$.

This action encodes all the information about the phenomenology of the bilayer JJA system. The electromagnetic response of the system is derived from the correlation functions $\delta^2 S/\delta A_\mu^{(\sigma)}(-q)\delta A_\nu^{(\sigma')}(q)$. In the following sections, we analyze the long wavelength $q \to 0$ and low frequency ω limits of these correlation functions for various condensates represented by integers n^α and m^α.

3.1. Condensate with $n_1 = n_2 = 1$ and $m_1 = m_2 = m$

In this case, the electric and magnetic excitations form bound states in each layer and the interlayer interaction forces them to form a composite condensate (see figure 2).

Using $\theta_1^- = \theta_2^- = 0$, the effective field theory contains a non-vanishing Chern–Simons term

$$S = -\frac{e^2}{2\pi m}\int A_\mu^+\varepsilon^{\mu\lambda\nu}q_\lambda A_\nu^+ + \frac{e^2}{4\pi^2\kappa_1}\int A_\mu^-\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)A_\nu^-. \tag{18}$$

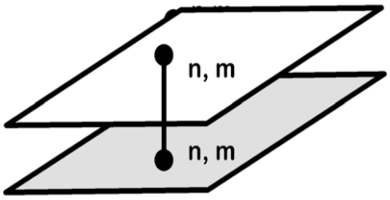

Figure 2. Schematic representation of the condensate with $n_1 = n_2 = 1$ and $m_1 = m_2 = m$.

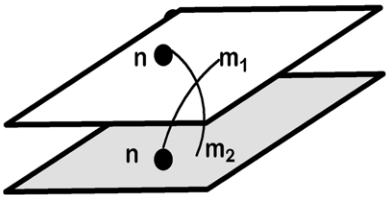

Figure 3. Schematic representation of the condensate with $n_1 = n_2 = 1$ and $m_1 \neq m_2$.

Varying with respect to $A_\mu^\pm$, we determine the electromagnetic currents $J_\mu^+ = \mathrm{i}e^2/(\pi m)\varepsilon^{\mu\lambda\nu}\partial_\lambda A_\nu^+$ and $J_\mu^- = e^2/(2\pi^2\kappa_1)A_\mu^-$. From the $\mu = \mathrm{i}$ component of the current, an electric field produces a current in the orthogonal direction with a quantized Hall conductivity in each layer $\sigma_{xy}^{11} = \sigma_{xy}^{22} = 1/4m$ in units of $4e^2/(2\pi\hbar)$ and also a quantized Hall drag conductivity $\sigma_{xy}^{12} = 1/4m$, namely, a Hall driving force in one layer induces a dragged current in the second layer. Furthermore, the second term in the above action shows that the system exhibits interlayer coherence. This can be interpreted as a perfect drag with currents along the two arrays equally large in magnitude but opposite in direction $\sigma_{xx}^{12}(\omega) = -e^2E_J/\omega$.

3.2. *Condensate with $n^- \neq 0$ or $m^- \neq 0$*

In this case the long wavelength limit $q \to 0$ and small frequency ω leads to the possibility of a Chern–Simons or of a Meissner term

$$S = \frac{2e^2}{\pi}n^+m^-K^-\int A_\mu^+\varepsilon^{\mu\lambda\nu}q_\lambda A_\nu^- + \frac{2e^2}{\pi}K^-\int A_\mu^-\left[\omega_J n^{-2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - n^+m^+m^{-2}\sqrt{\kappa_2/\kappa_1}K^-\varepsilon^{\mu\lambda\nu}q_\lambda\right]A_\nu^-, \tag{19}$$

where $K^- = 1/[n^{-2}\sqrt{\kappa_1/\kappa_2} + m^{-2}\sqrt{\kappa_2/\kappa_1}]$. Below we analyze special cases.

3.2.1. *Condensate with $n^- = 0$ and $m^- \neq 0$.*

In this state ($n_1 = n_2 = 1$), charges from one layer bind to vortices from the other layer and the two composites condense (see figure 3). The state thus formed has a quantized Hall conductance without interlayer coherence

$$S = -\frac{e^2}{2\pi}K_{\alpha\alpha'}\int A_\mu^{(\alpha)}\varepsilon^{\mu\lambda\nu}q_\lambda A_\nu^{(\alpha')} + \text{Maxwell term}.$$

$$K_{\alpha\alpha'} = 8n/(m_1 - m_2)^2\begin{bmatrix} m_2 & -(m_1+m_2)/2 \\ -(m_1+m_2)/2 & m_1\end{bmatrix}.$$

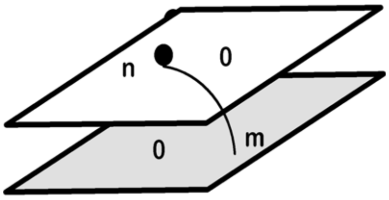

Figure 4. Schematic representation of a bound state of an electric excitation ($n = 1$) from one array and a magnetic excitation (m) from the other array.

As a special case, $m_1 = m$ and $m_2 = -m$, we find a quantized Hall conductivity in each layer $\sigma_{xy}^{11} = -\sigma_{xy}^{22} = 1/4m$ in units of $4e^2/(2\pi\hbar)$.

3.2.2. Condensate with $n^- \neq 0$ and $m^- = 0$. In this case the Chern–Simons term vanishes and the effective dynamics of the external electromagnetic gauge potential has a Meissner-type term

$$S_g = \frac{e^2}{\pi^2\kappa_1}\int A_\mu^-\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)A_\nu^-. \tag{20}$$

In this state, the quantum Hall effect is destroyed without destroying interlayer coherence.

3.2.3. Condensate with $n^- = n^+, m^- = -m^+$. The condensate formed consists of n electric charges from one array forming a bound state with m magnetic charges from the other array (see figure 4).

The effective dynamics of the external electromagnetic gauge potential has a non-vanishing Chern–Simons term and a Meissner term

$$\begin{aligned} S_g = -\frac{2me^2}{\pi}K\int\left[A_\mu^+\varepsilon^{\mu\lambda\nu}q_\lambda A_\nu^- + Km^2\sqrt{\kappa_2/\kappa_1}A_\mu^-\varepsilon^{\mu\lambda\nu}q_\lambda A_\nu^-\right] \\ + \frac{2e^2\omega_J}{\pi}K\int A_\mu^-\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)A_\nu^-, \end{aligned} \tag{21}$$

where $K = 1/[\sqrt{\kappa_1/\kappa_2} + m^2\sqrt{\kappa_2/\kappa_1}]$. In this case the Hall state in the bilayer JJA system coexists with interlayer coherence. At the self-dual point ($\kappa_1 = \kappa_2$), the Hall conductance is quantized.

4. Conclusion

In this paper, I have presented results of a model of two coupled layers of JJA frustrated by external offset charges and magnetic fields. The coupling between the layers was imposed by requiring self-duality between the charges and vortices. Such an approximation is valid at low energies compared to relevant energy scales of the system ($E \ll E_C, E_J$) since the duality-breaking terms are suppressed in that energy regime. This allowed a suitable gauge theory representation of the model of interacting charges and vortices, and generated in the bilayer JJA system the most general interactions between the charges and the vortices. Starting from the Abelian gauge theory with a mixed Chern–Simons term and using duality, I showed that the fundamental constituents of this theory are electric and magnetic excitations and their condensation leads to a variety of possible phases. In addition to the usual superconducting and insulating states in the bilayer system, I find states exhibiting Hall quantization coexisting with interlayer coherence, states with Hall quantization without interlayer coherence and interlayer coherent states without Hall quantization. In our approach, the Hall states originate from the peculiar coupling between the electric-magnetic excitations and the gauge fields, which in effect attaches an even number of fictitious vortices to each Cooper pair and correspondingly by self-duality of the model attaches an even number of fictitious Cooper pairs to each vortex. These results, not attainable by standard mean field theory approaches, are significant since they imply the realization of bosonic quantized Hall states analogous to the fractional quantum Hall effect in semiconductor heterojunctions, which opens new fundamental theoretical understandings, and the detection of the various states presents an interesting experimental challenge in its own right. For the most part the experimentally fabricated Josephson junction systems have been two dimensional, but prototype quasi-three-dimensional samples have also been fabricated [11]. The insulating–superconducting quantum phase transition is actually observed in JJA at low temperatures [1]. As for the experimental work on the Hall effect in JJA, so far no conclusive results for the Hall quantization in these systems are known. However the Hall measurements of [12] indicate that the Hall effect in these systems is more complicated and exhibits some interesting characteristics such as a periodic Hall resistance with respect to the applied magnetic field and a larger Hall angle consistent with the expectation that the offset charges are responsible for the Hall effect. Our approach analyzes the quantum Hall state by means of an effective field theory describing vortex–charge bound states. As such, it cannot determine the exact range of parameters under which the quantum Hall state is the ground state. Below we make some qualitative comments regarding that question. The observation of a Hall voltage in JJA requires very low temperature (to suppress thermal fluctuations) and the parameters of the array should be such that a balance is achieved between the charging energy and the Josephson energy $E_C \approx E_J$. This is crucial since in a strongly superconducting array ($E_J \gg E_C$) any Hall probes would be shorted leading to zero Hall voltage. Whereas in an array with a strong Coulomb blockade ($E_C \gg E_J$) the whole array is insulating and therefore the Hall probes are effectively disconnected, and no Hall voltage can be measured.

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